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On the Second Order Local Comparison between Perturbed Maximum Likelihood Estimators and Rao's Statistic as Test Statistics

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A bias-adjusted maximum likelihood estimator (mle), which has been shown to possess certain optimality criteria as an estimate of θ (see Ghosh, J. K., Sinha, B. K., and Joshi, S. N. (1982). In *Statistical Decision Theory and Related Topics III* (S. S. Gupta and J. O. Berger, Eds.), Vol. 1, pp. 403–456. Academic Press, Orlando, FL) is compared with Rao's statistic as a test statistic for the standard two-sided testing problem. It is shown that Rao's statistic is locally superior to any bias-adjusted mle in the sense of Chandra and Joshi (1983, *Sankhyā Ser. A* **45** 226–246). A second interpretation of a conjecture of C. R. Rao is proposed and Rao's statistic is shown to be superior as a test statistic according to the new interpretation as well. The last fact provides an interesting supplement to the results of Chandra and Joshi (1983, *Sankhyā Ser. A* **45** 226–246). Furthermore, a partial reason for the inferiority of the likelihood ratio and Wald's statistic to Rao's statistic is supplied and certain regularity assumptions of the last paper are eliminated. Finally, the local powers of certain modified versions of Rao's and Wald's statistics (see Skovgaard, I. M. (1985). *Ann. Statist.* **13** 534–551) are studied.

1. INTRODUCTION AND DEFINITIONS

Let $\{X_n\}$ be a sequence of iid random vectors with a common density $f(x;\theta)$, where θ takes values in a non-empty open subset of the real line; we shall assume that the support of $f(x;\theta)$ is free from θ. Consider the problem of testing $H_0: \theta=\theta_0$ versus $H_1: \theta\neq\theta_0$; we shall assume that $\theta_0=0$. Let

$$\lambda_n^1 = 2\left(\sum \log f(x_i;\hat\theta) - \sum \log f(x_i;\theta_0)\right),$$
$$\lambda_n^2 = (nI)^{-1}\left(\sum D\log f(x_i;\theta_0)\right)^2,$$
$$\lambda_n^3 = n(\hat\theta-\theta_0)^2 I_n;$$

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here $\hat{\theta}$ is the maximum likelihood estimator (mle) of θ based on the first n observations $(x_1, ..., x_n)$, D is the operator $\partial/\partial\theta$ and

$$I(\theta) = E_\theta(D \log f(X_1; \theta))^2, \qquad I = I(\theta_0),\, I_n = I(\hat{\theta}).$$

Assume that E_{θ_0} $(D \log f(X_1;\ \theta_0)) = 0$, $0 < I < \infty$. For the above testing problem, three commonly used tests are based on the critical regions

$$\{\lambda_n^i > \chi^2_{1,\alpha}\}, \qquad i = 1, 2, 3, \tag{1.1}$$

where α denotes the (approximate) level of significance, $0 < \alpha < 1$, and $\chi^2_{1,\alpha}$ is the upper α-point of the chi-square distribution with 1 degree of freedom. The statistics λ_n^i, $i = 1, 2, 3$, have been proposed by Neyman and Pearson (1928), Rao (1948), and Wald (1943), respectively (see [11, pp. 347–352]). It has been shown by Chandra and Joshi [3] that (i) it is possible to get "asymptotically locally unbiased" (up to $o(n^{-1})$) versions of the critical regions (1.1) (by introducing "perturbations" of the boundaries of these regions) whose sizes are comparable up to $o(n^{-1})$ and whose local powers agree up to $o(n^{-1/2})$ and (ii) the power of the modified Rao's test is "locally" larger than those of the other two (modified) tests for reasonable values of α; hence the conclusion of Peers [9] (see also [7]) that the local powers of the critical regions (1.1) are not comparable up to $o(n^{-1/2})$ is of little interest. Chandra and Mukerjee [4] have extended (ii) by showing that the modified Rao's test has larger local power than those of a certain family of tests. *In this paper we shall follow* [3] *in choosing the cut-off points of the test statistics.*

Since it is well known that a bias-adjusted mle is a "good" estimator of θ (for precise statements, see [6] and its references), it is worthwhile to investigate the relative performance of a bias-adjusted mle and (the square root of) Rao's statistic as test statistics in the sense of [3]. Since a bias-adjusted mle is of the form

$$\hat{\theta} + n^{-1/2} t_1(\hat{\theta}) + n^{-1} t_2(\hat{\theta}), \tag{1.2}$$

where t_1 and t_2 are suitable real-valued functions, we consider here the family of all statistics of the form (1.2); we shall assume that the functions t_i are sufficiently smooth in a neighbourhood of θ_0. Consider now the set A_n described in (3.2) [3, p. 236]; since it is valid to expand each $t_i(\hat{\theta})$ around θ_0 in the Taylor series on A_n, we need only compare

$$\hat{\theta}^* + n^{-1/2} e_1 \hat{\theta}^* + n^{-1}(e_2 \hat{\theta}^* + e_3(\hat{\theta}^*)^2),\ (\hat{\theta}^* = (nI)^{1/2}\, \hat{\theta}) \tag{1.3}$$

with Rao's statistic, where the e_i are arbitrary constants free from n (statistics of the form (1.3) will be called perturbed mle's); using the expansion of $n^{1/2}\hat{\theta}$ given on p. 236 of [3] with $\delta = 0$, it is easy to verify that we are led to the following family $\mathscr{F}$ of the sequence $\{\lambda_n\}$ of statistics, where $v_1 = I^{-3/2}$, $v_3 = v_5 = 0$ (recall that $v_1 \neq 0$ for all statistics of the form λ_n^1, λ_n^3). *For each* $\{\lambda_n\}$ *in* $\mathscr{F}$, *there exist a sequence* $\{B_n\}$ *of sets and a sequence* $\{W_n\}$ *of statistics such that*

$$P_{\theta_n}(B_n) = 1 + o(n^{-1}) \qquad \text{for each real } \delta, \tag{1.4}$$

$$\lambda_n = (W_n)^2 + o(n^{-1}) \qquad \text{on} \quad B_n, \tag{1.5}$$

where $\theta_n = \theta_0 + \delta n^{-1/2}$ *and*

$$\begin{aligned} W_n = H_1 I^{-1/2} &+ n^{-1/2}[v_1 H_1 H_2 + v_2 H_1^2 + v_3 H_2^2 \\ &+ v_4 H_1 + v_5 H_2] + n^{-1}[y_1 H_1 H_2^2 + y_2 H_1^2 H_2 \\ &+ y_3 H_1^3 + y_4 H_1^2 H_3 + y_5 H_1 H_3^2 + y_6 H_2^3 \\ &+ y_7 H_1 H_2 + y_8 H_1^2 + y_9 H_2^2 + y_{10} H_1 + y_{11} H_2], \end{aligned} \tag{1.6}$$

$$H_i = n^{-1/2}\left(\sum_j D^i \log f(x_j; \theta_0) - n m_i\right), \tag{1.7}$$

$$m_i(\theta) = E_\theta(D^i \log f(x_1; \theta)), \qquad m_i = m_i(\theta_0) \tag{1.8}$$

$(i = 1, 2, 3)$, *the* v_i *and* y_i *being arbitrary constants which are free from* n, α *and* δ. It will be shown that the inclusion of the terms involving v_3 and v_5 in (1.6) enables one to get a wide class of "asymptotically locally unbiased" (up to $o(n^{-1})$) tests with large local powers; it can be shown that the local power does not depend on the terms of order n^{-1} even if one replaces the third term of the right side of (1.6) by

$$n^{-1} \sum y_{i,j,k} H_1^i H_2^j H_3^k,$$

where the summation is taken over all non-negative integers i, j, k such that $i + j + k \leqslant 3$. It should be noted that the above family $\mathscr{F}$ of statistics has the desired property of being closed under perturbations of the form (1.3). *It is clear that the desired comparison cannot be made* from the results of [3, 4]; *moreover, the important facts* (A) *to* (F) *of* [3] (*see pp.* 233–235) *and certain identities of Section* 4 *of* [3] *do not necessarily hold for the above family of tests.* It follows from Section 3 that the modified Rao's test is locally superior (in the sense of [3]) to any test based on a statistic in $\mathscr{F}$ and that there is no test based on such statistics which is superior to Rao's test for all values of δ, the common size of the tests being held fixed. In fact, for each real δ, we have shown that

$$\begin{aligned} P_2^2 - P_2 = \tfrac{1}{2}\,|\delta|\, I^{1/2}\phi(z - |\delta|\, I^{1/2})[&I^2\gamma_{\theta_0}^2 B_1 \\ &+ |\delta|\, I^{-1/2} C_1 B_3 + I^2\gamma_{\theta_0}^2(1 - B_5)\, B_6 \\ &- B_5(I^2\gamma_{\theta_0}^2 B_2 - |\delta|\, I^{-1/2} C_1 B_4)], \end{aligned} \tag{1.9}$$

provided certain regularity conditions hold (see Section 3); here P_2^2, P_2 are respectively the coefficients of n^{-1} in the powers under θ_n of the (both-sided) tests based on the square roots of λ_n^2 and λ_n,

$$V = v_1 + 2L_{11} v_3 I^{-1} \tag{1.10}$$

$$\begin{aligned} &B_3 = VzI^{1/2} + \operatorname{sign}(\delta)\, v_5, \qquad &&B_4 = VzI^{1/2} - \operatorname{sign}(\delta)\, v_5, \\ &B_1 = (B_3 - \tfrac{1}{2}|\delta|\, I^{-1/2})^2, &&B_2 = (B_4 + \tfrac{1}{2}|\delta|\, I^{-1/2})^2, \\ &B_5 = \exp(-2z\,|\delta|\, I^{1/2}), &&B_6 = 2I^2\gamma_{\theta_0}^2 v_3^2 - \tfrac{1}{4}\delta^2 I^{-1}, \end{aligned} \tag{1.11}$$

$$\begin{aligned} C &= L_{001} + 3L_{11} + L_3, \\ C_1 &= CL_{11} I^{-1} + m_{22}'' - L_{0001} - 2L_{101} - L_{21} - L_{02}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} &L_{ijkt} = E_{\theta_0}[h_1^i h_2^j h_3^k h_4^t], \\ &\quad h_i = D^i \log f(X, \theta_0), \qquad i = 1, 2, 3, 4, \\ &L_{ijk} = L_{ijk0}, \qquad L_{ij} = L_{ij0}, \qquad L_i = L_{i0}, \\ &\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2), \\ &\quad \gamma_{\theta_0} = [I(L_{02} - I^2) - L_{11}^2]^{1/2}\, I^{-3/2}, \end{aligned}$$

z is the upper $(\alpha/2)$-point of a normal deviate and primes stand for differentiation with respect to θ at θ_0. It follows that "optimal" test statistics in $\mathscr{F}$ must satisfy (under the assumption that $\gamma_{\theta_0} > 0$)

$$V = 0,\ v_3 = v_5 = 0 \qquad \text{if} \quad C_1 = 0; \tag{1.13}$$

then the powers of the tests based on λ_n and λ_n^2 are identical up to $o(n^{-1})$. *The condition that* $V = 0$ *is violated by the statistics* λ_n^1, λ_n^3 *and their analogues discussed in Section* 4; *we have thus shown a partial reason for the inferiority of the likelihood ratio and Wald's statistics to Rao's statistics*; if $v_1 \neq 0$, then one must take v_3 so that $V = 0$. *Furthermore, the test statistics for which* V *vanishes are "asymptotically locally unbiased" up to* $o(n^{-1})$ *and have local powers* larger than the statistics of the form λ_n^1, λ_n^3 *for all sufficiently small* α; *also these statistics lie outside the family of statistics considered by* [4] *unless* $v_3 = v_4 = v_5 = 0$. Condition (1.13) is equivalent to requiring that W_n depend only on H_1 up to terms of order $(n^{-1/2})$. One may note that if $\gamma_{\theta_0}=0$, then the powers of both the tests are identical up to $o(n^{-1})$, provided $C_1=0$.

To make the notion of "optimality" precise, we shall first fix some notation and definition. Let W_n^2 *be the square root of* λ_n^2, i.e.,

$$W_n^2=(nI)^{-1/2}\sum D\log f(x_i;\theta_0). \tag{1.14}$$

If g is a polynomial of a single variable x and r is a non-negative integer, then $g(r)$ *will stand for the coefficient of* x^r *in* $g(x)$. Let $\{\lambda_n\}$ be a sequence of statistics such that there exist a sequence $\{B_n\}$ of sets and a sequence W_n of statistics satisfying (1.4), (1.5) and the following conditions (C1) and (C2).

(C1) *There exist constants* b_i, c_i, b_i^2, c_i^2, d_i *and* e_i for $i=1,2$ *such that each is free from n and* δ, *the* d_i *are polynomials in z of degree at most 2 and for each real* δ *we have*

$$\begin{aligned}
&P_{\theta_n}(W_n>z+n^{-1/2}b_1+n^{-1}c_1)\\
&\quad=P_{\theta_n}(W_n^2>z+n^{-1/2}b_1^2+n^{-1}c_1^2)\\
&\quad=\int_{z-\delta I^{1/2}}^{\infty}\phi(t)\,dt+\tfrac{1}{6}n^{-1/2}\phi(z-\delta I^{1/2})\\
&\qquad\times[6d_1+\delta z(L_3-3C)\,I^{-1}+\delta^2(3L_{11}+L_3)\,I^{-1/2}]\\
&\qquad+n^{-1}\phi(z-\delta I^{1/2})[e_1+O(\delta)]+o(n^{-1}),
\end{aligned}\tag{1.15}$$

$$\begin{aligned}
&P_{\theta_n}(W_n<-z+n^{-1/2}b_2+n^{-1}c_2)\\
&\quad=P_{\theta_n}(W_n^2<-z+n^{-1/2}b_2^2+n^{-1}c_2^2)\\
&\quad=\int_{z+\delta I^{1/2}}^{\infty}\phi(t)\,dt+\tfrac{1}{6}n^{-1/2}\,\phi(z+\delta I^{1/2})\\
&\qquad\times[-6d_2+\delta z(L_3-3C)\,I^{-1}-\delta^2(3L_{11}+L_3)\,I^{-1/2}]\\
&\qquad+n^{-1}\phi(z+\delta I^{1/2})[e_2+O(\delta)]+o(n^{-1}).
\end{aligned}\tag{1.16}$$

Remark 1.1. The requirement that *the* d_i *be polynomials in z of degree at most* 2 comes from the fact that the coefficient of $n^{-1/2}$ of the Edgeworth expansion under θ_0 of a statistic like W_n is ($\phi(z)$ times) a polynomial of degree at most 2; from this point of view, one needs only consider the special case

$$d_1(j)=d_2(j)\quad\text{for}\quad j=0,2,\qquad d_1(1)=-d_2(1). \tag{1.17}$$

Chandra and Joshi [3] consider the further special case when $d_i(j)=0$ for $i=1,2$ and $j=1,2$. We treat here the general case since the power of many interesting tests can then be obtained from Sections 2 and 3.

To state condition (C2), denote the power at θ_n of the both-sided test based on W_n by

$$P_{n,\delta,\alpha} = P_{\theta_n}(W_n > z + n^{-1/2}b_1 + n^{-1}c_1 \text{ or } < -z + n^{-1/2}b_2 + n^{-1}c_2)$$
$$= P_0 + n^{-1/2}P_1 + n^{-1}P_2(\delta, \alpha) + o(n^{-1}), \tag{1.18}$$

say (here P_0, P_1 and $P_2(\delta, \alpha)$ are free from n); define, similarly, $P^2_{n,\delta,\alpha}$ and $P^2_2(\delta, \alpha)$ by replacing W_n by W^2_n in (1.18).

(C2) *The expression*

$$S_\alpha = \partial^2(P^2_2(\delta, \alpha) - P_2(\delta, \alpha))/\partial\delta^2 \qquad at \quad \delta = 0$$

is $\phi(z)$ times an odd polynomial in z of degree at most 3. We say that Rao's statistic is locally superior to λ_n as a test statistic if for all sufficiently small α,

$$\lim_{\delta \to 0} (P^2_2(\delta, \alpha) - P_2(\delta, \alpha))/\delta^2 \geqslant 0. \tag{1.19}$$

We say that the statistic λ_n is inferior to Rao's statistic in the sense (I) *if*

$$\lim_{\alpha \to 0} (z^3\phi(z))^{-1} S_\alpha > 0, \tag{1.20}$$

and it is inferior to Rao's statistic in the sense (II) *if the left side of Inequality* (1.20) *is zero and*

$$S_\alpha \geqslant 0 \qquad \textit{for all } \alpha. \tag{1.21}$$

It follows from [3] that λ^1_n and λ^3_n are inferior to Rao's statistic in the sense (I), provided certain regularity conditions hold and the Efron curvature at θ_0, γ_{θ_0}, is positive (see also Remark 1.2, below); the same conclusion holds for a statistic which belongs to the family considered by [4] and has $v_1 \neq 0$. The statistics for which the V vanish are, however, equivalent to Rao's statistic in the sense (I) (i.e., the left side of Inequality (1.20) is zero); they are inferior to Rao's statistic in the sense (II), provided $C_1 = 0$.

Remark 1.2. If Rao's statistic is locally superior to λ_n as a test statistic, then

$$\partial(P^2_2(\delta, \alpha) - P_2(\delta, \alpha))/\partial\delta \text{ vanishes at } \delta = 0. \tag{1.22}$$

If (1.22) holds, the function $(P^2_2(\delta, \alpha) - P_2(\delta, \alpha))$ is smooth in a neighbourhood of $\delta = 0$ and λ_n is inferior to Rao's statistic either in the sense (I) or in the sense (II); then Rao's statistic will be locally superior to λ_n as a test statistic. Condition (1.22) is to be regarded as a generalisation of the asymptotic local unbiasedness (we say that the both-sided test based on W_n is asymptotically locally unbiased up to $o(n^{-1})$ if $\partial P_1/\partial\delta$ and $\partial P_2(\delta, \alpha)/\partial\delta$ are zero at $\delta = 0$). If (1.17) holds d_1, d_2 are free from α, then $\partial P_1/\partial\delta$ at $\delta = 0$ vanishes iff $d_1 = -\frac{1}{6}(L_3 - 3C)\, I^{-3/2}$.

We now give another interpretation of Rao's conjecture that λ_n^2 is likely to be locally superior to λ_n^1 and λ_n^3 (see [10, 11]). *We say that the statistic λ_n is inferior to Rao's statistic in the sense* (III) *if for each non-zero δ,*

$$\lim_{\alpha \to 0} (P_2^2(\delta, \alpha) - P_2(\delta, \alpha))/(z^2 \phi(z - |\delta| I^{1/2})) > 0, \tag{1.23}$$

and it is inferior to Rao's statistic in the sense (IV) *if the left side of Inequality* (1.23) *is zero and*

$$\lim_{\delta \to 0} \lim_{\alpha \to 0} (P_2^2(\delta, \alpha) - P_2(\delta, \alpha))/(|\delta| \phi(x - |\delta| I^{1/2})) > 0. \tag{1.24}$$

It should be emphasized that unlike [3], the last two definitions do not require that the both-sided tests based on W_n and W_n^2 are asymptotically locally unbiased up to $o(n^{-1})$. It is shown in Section 3 that if $V \neq 0$, then λ_n (in $\mathscr{F}$) is inferior to Rao's statistic in the sense (III); *in particular, Rao's conjecture is true according to the new interpretation.* If $V=0$, $C_1=0$ and either v_3 or v_5 is non-zero, then λ_n is inferior to Rao's statistic in the sense (IV). *Thus the failure of condition* (1.13) *is equivalent to the inferiority of λ_n to Rao's statistic as a test statistic according to the second optimality* criterion.

2. Some General Results

Let $\{\lambda_n\}$ be a sequence of statistics such that there exist a sequence $\{B_n\}$ of sets, a sequence $\{W_n\}$ of statistics and polynomials k_{ij} ($i=1,2,3,4$; $j=1,2$) *of a single variable δ* (*free from n and α*) *satisfying* (1.4), (1.5) *and Eqs.* (2.1) *to* (2.6) *below.*

$$\begin{aligned} P_{\theta_n}(W_n - \delta I^{1/2} > a + n^{-1/2} b + n^{-1} c) \\ = \int_a^\infty \phi(t)\, dt + n^{-1/2}\phi(a)\left[-b + \sum_{r=1}^{3} J_{r-1}(a)\, k_{r1}/r!\right] \\ + n^{-1}\phi(a)\Bigg[-c + \tfrac{1}{2}ab^2 - b \sum_{r=1}^{3} J_r(a)\, k_{r1}/r! \\ + k_{12} + \tfrac{1}{2}J_1(a)(k_{22} + (k_{11})^2) \\ + \tfrac{1}{6}J_2(a)(k_{32} + 3k_{21}k_{11}) \\ + \tfrac{1}{24}J_3(a)(k_{42} + 3(k_{21})^2 + 4k_{11}k_{31}) \\ + \tfrac{1}{12}J_4(a)\, K_{21}k_{31} + \tfrac{1}{72}J_5(a)(k_{31})^2\Bigg] \\ + o(n^{-1}), \qquad \text{for each } \delta \text{ and constants, } a, b, c \text{ free from } n. \end{aligned} \tag{2.1}$$

$$2k_{11}(1) = k_{21}(0)\, I^{1/2}, \tag{2.2}$$

$$6k_{11}(2) = (3L_{11} + 2L_3 - 3C)\, I^{-1/2} + k_{31}(0)\, I, \tag{2.3}$$

$$3k_{21}(1) = (L_3 - 3C)\, I^{-1} + 2k_{31}(0)\, I^{1/2}, \tag{2.4}$$

$$k_{11}(r) = 0 \quad \text{if} \quad r \geqslant 3; \qquad k_{21}(r) = 0 \quad \text{if} \quad r \geqslant 2;$$

$$k_{31}(r) = 0 \quad \text{if} \quad r \geqslant 1, \tag{2.5}$$

$$k_{42}(1) = k_{42}(2) = 0. \tag{2.6}$$

In (2.1), the symbol J_r denotes the Hermite polynomial of order r, i.e., for each real t,

$$J_r(t)\, \phi(t) = (-1)^r\, d^r\phi(t)/dt^r, \qquad r \geqslant 1, J_0 = 1. \tag{2.7}$$

It will be assumed that Eqs (2.1) *to* (2.6) *hold with* W_n *and* k_{ij} *replaced by* W_n^2 and k_{ij}^2, respectively (following [3], we adopt the convention that any upper suffix i will indicate that W_n is replaced by W_n^i for $i \geqslant 1$). Thus results about W_n^2 can be obtained from those about W_n.

Now let d_i, e_j $(i = 1, ..., 6; j = 1, 2)$ *be fixed constants free from n and* δ; *the* d_i *are, moreover, free from* α. Let the constants b_i, c_i, b_i^2, c_i^2 $(i = 1, 2)$ be chosen so that they are free from n and

$$\begin{aligned}
&P_{\theta_0}(W_n > z + n^{-1/2}b_1 + n^{-1}c_1)\\
&\quad = P_{\theta_0}(W_n^2 > z + n^{-1/2}b_1^2 + n^{-1}c_1^2)\\
&\quad = \tfrac{1}{2}\alpha + \phi(z)[n^{-1/2}(d_1 + d_2 z + d_3 z^2) + n^{-1}e_1] + o(n^{-1}),\\
&P_{\theta_0}(W_n < -z + n^{-1/2}b_2 + n^{-1}c_2)\\
&\quad = P_{\theta_0}(W_n^2 < -z + n^{-1/2}b_2^2 + n^{-1}c_2^2)\\
&\quad = \tfrac{1}{2}\alpha + \phi(z)[n^{-1/2}(-d_4 + d_5 z - d_6 z^2) + n^{-1}e_2] + o(n^{-1}).
\end{aligned}$$

It then follows from (2.1) (with $\delta = 0$) that b_1 and b_2 are polynomials in z of degree at most 2 and that

$$\begin{aligned}
b_1(0) &= -d_1 + k_{11}(0) - \tfrac{1}{6}k_{31}(0),\\
b_1(1) &= \tfrac{1}{2}k_{21}(0) - d_2,\\
b_1(2) &= \tfrac{1}{6}k_{31}(0) - d_3;
\end{aligned} \tag{2.8}$$

the expression of b_2 can be obtained from that of b_1 by replacing z, d_1, d_2, d_3 by $-z$, d_4, d_5 and d_6, respectively. *Equations* (2.1) *to* (2.4) *then imply that condition* (C1) *of Section* 1 *holds.* Let the coefficient of n^{-1} in $P_{\theta_n}(W_n > z + n^{-1/2}b_1 + n^{-1}c_1)$ be denoted by $\phi(z - \delta I^{1/2})(e_1 + \delta Q_1 +$

$\delta^2 Q_2 + O(\delta^3))$, where Q_1 and Q_2 are free from δ and are polynomials in z. It is immediate from (2.1), (2.5) and (2.6) that

$$\begin{aligned}
Q_1 = & -\tfrac{1}{2}(b_1)^2 I^{1/2} + b_1[k_{11}(0) I^{1/2} - zk_{11}(1) + zk_{21}(0) I^{1/2} \\
& + \tfrac{1}{2}(1-z^2)(k_{21}(1) - k_{31}(0) I^{1/2})] + k_{12}(1) - \tfrac{1}{2}(k_{22}(0) \\
& + (k_{11}(0))^2) I^{1/2} + z[\tfrac{1}{2}k_{22}(1) + k_{11}(0)\, k_{11}(1) - (\tfrac{1}{3}k_{32}(0) \\
& + k_{21}(0)\, k_{11}(0)) I^{1/2}] + \tfrac{1}{24}J_2(z)[4k_{32}(1) + 12k_{21}(1)\, k_{11}(0) \\
& + 12k_{21}(0)\, k_{11}(1) - (3k_{42}(0) + 12k_{11}(0)\, k_{31}(0) \\
& + 9(k_{21}(0))^2) I^{1/2}] + \tfrac{1}{12}J_3(z)[3k_{21}(0)\, k_{21}(1) \\
& + 2k_{11}(1)\, k_{31}(0) - 4k_{21}(0)\, k_{31}(0) I^{1/2}] + \tfrac{1}{72}J_4(z) \\
& \times (6k_{21}(1)\, k_{31}(0) - 5(k_{31}(0))^2 I^{1/2}), \\
Q_2 = & \; zb_1[-k_{11}(2) + k_{21}(1) I^{1/2} - \tfrac{1}{2}k_{31}(0) I] \\
& + b_1[k_{11}(1) I^{1/2} - \tfrac{1}{2}k_{21}(0) I] + k_{12}(2) \\
& + z[\tfrac{1}{2}k_{22}(2) + \tfrac{1}{2}(k_{11}(1))^2 + k_{11}(0)\, k_{11}(2) \\
& - (\tfrac{1}{3}k_{32}(1) + k_{21}(0)\, k_{11}(1) + k_{21}(1)\, k_{11}(0)) I^{1/2} \\
& + \tfrac{1}{8}(k_{42}(0) + 3(k_{21}(0))^2 + 4k_{11}(0)\, k_{31}(0) I)] \\
& - (\tfrac{1}{2}k_{22}(1) + k_{11}(0)\, k_{11}(1)) I^{1/2} + \tfrac{1}{6}(k_{32}(0) + 3k_{21}(0)\, k_{11}(0)) I \\
& + \tfrac{1}{2}J_2(z)[k_{21}(1)\, k_{11}(1) + k_{21}(0)\, k_{11}(2) - \tfrac{3}{2}k_{21}(0)\, k_{21}(1) I^{1/2} \\
& - k_{11}(1)\, k_{31}(0) I^{1/2} + k_{21}(0)\, k_{31}(0) I].
\end{aligned} \tag{2.9}$$

One can verify that Q_1 and Q_2 are polynomials of degree at most 4 and 3, respectively, and that

$$\begin{aligned}
Q_1(0) = & \; k_{12}(1) - \tfrac{1}{6}k_{32}(1) + [\tfrac{1}{8}k_{42}(0) + \tfrac{1}{8}(k_{21}(0))^2 \\
& - \tfrac{1}{2}k_{22}(0) - \tfrac{1}{36}(k_{31}(0))^2 - \tfrac{1}{2}d_1^2] I^{1/2} \\
& + \tfrac{1}{18}(k_{31}(0) - 3d_1)(L_3 - 3C) I^{-1}, \\
Q_1(1) = & \; \tfrac{1}{6}[k_{31}(0)\, k_{21}(0) - 2k_{32}(0) - 6d_1 d_2] I^{1/2} \\
& + \tfrac{1}{2}k_{22}(1) - \tfrac{1}{6}(k_{21}(0) + d_2)(L_3 - 3C) I^{-1}, \\
Q_1(2) = & \; \tfrac{1}{6}k_{32}(1) - \tfrac{1}{8}k_{42}(0) I^{1/2} + \tfrac{1}{18}(k_{31}(0))^2 I^{1/2} \\
& - \tfrac{1}{9}k_{31}(0)(L_3 - 3C) I^{-1} - \tfrac{1}{2}(d_2^2 + 2d_1 d_3) I^{1/2} \\
& + \tfrac{1}{6}(d_1 - d_3)(L_3 - 3C) I^{-1},
\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
Q_1(3) &= -d_2 d_3 I^{1/2} + \tfrac{1}{6} d_2 (L_3 - 3C)\, I^{-1},\\
Q_1(4) &= -\tfrac{1}{2} d_3^2 I^{1/2} + \tfrac{1}{6} d_3 (L_3 - 3C)\, I^{-1},\\
Q_2(0) &= k_{12}(2) - \tfrac{1}{2} k_{22}(1)\, I^{1/2} + \tfrac{1}{6} k_{32}(0)\, I - \tfrac{1}{4} k_{21}(0)(L_{11} + C)\, I^{-1/2}\\
Q_2(1) &= \tfrac{1}{2} k_{22}(2) - \tfrac{1}{3} k_{32}(1)\, I^{1/2} + \tfrac{1}{8} k_{42}(0)\, I - \tfrac{1}{6} k_{31}(0)(L_{11} + C)\, I^{-1/2}\\
&\quad - \tfrac{1}{24}(L_3 - 3C)^2 I^{-2} + \tfrac{1}{2}(L_{11} + C)\, d_1 I^{-1/2}, \qquad (2.11)\\
Q_2(2) &= \tfrac{1}{2}(L_{11} + C)\, d_2 I^{-1/2},\\
Q_2(3) &= \frac{1}{72}(L_3 - 3C)^2 I^{-2} + \tfrac{1}{2}(L_{11} + C)\, d_3 I^{-1/2}.
\end{aligned}
$$

By replacing z^r by $(-1)^{r+1} z^r$ for $r \geqslant 0$ and then replacing d_1, d_2, d_3 by d_4, d_5 and d_6, respectively, one can obtain the coefficient of n^{-1} in $P_{\theta_n}(W_n < -z + n^{-1/2} b_2 + n^{-1} c_2)$. Hence we get

$$
\begin{aligned}
P_2(\delta, \alpha) = \phi(z) \exp(-\tfrac{1}{2}\delta^2 I)[e_1 \exp(\delta z I^{1/2})\\
+ e_2 \exp(-\delta z I^{1/2}) + \delta A_1 + \delta^2 A_2 + O(\delta^3)], \qquad (2.12)
\end{aligned}
$$

where the term $O(\delta^3)$ is a power series in δ and A_1, A_2 are polynomials in z of degree at most 4 and 5, respectively (see also (1.18)); moreover,

$$
\begin{aligned}
A_1(0) &= \tfrac{1}{6}(d_4 - d_1)[3(d_4 + d_6)\, I^{1/2} + (L_3 - 3C)\, I^{-1}],\\
A_1(1) &= (\tfrac{1}{3} k_{31}(0)\, k_{21}(0) - \tfrac{2}{3} k_{32}(0) - d_1 d_2 - d_4 d_5)^{1/2}\\
&\quad + k_{22}(1) - \tfrac{1}{6}(2k_{21}(0) + d_2 + d_5)(L_3 - 3C)\, I^{-1},\\
A_1(2) &= \tfrac{1}{2}(d_5^2 + 2 d_4 d_6 - d_2^2 - 2 d_1 d_3)\, I^{1/2}\\
&\quad + \tfrac{1}{6}(d_1 + d_6 - d_3 - d_4)(L_3 - 3C)\, I^{-1},\\
A_1(3) &= -(d_2 d_3 + d_5 d_6)\, I^{1/2} + \tfrac{1}{6}(d_2 + d_5)(L_3 - 3C)\, I^{-1},\\
A_1(4) &= \tfrac{1}{6}(d_6 - d_3)[3(d_3 + d_6)\, I^{1/2} - (L_3 - 3C)\, I^{-1}], \qquad (2.13)\\
A_2(0) &= 0,\\
A_2(1) &= k_{22}(2) + (2k_{12}(1) - k_{32}(1))\, I^{1/2} + [\tfrac{1}{2} k_{42}(0)\\
&\quad + \tfrac{1}{4}(k_{21}(0))^2 - k_{22}(0) - \tfrac{1}{18}(k_{31}(0))^2 - \tfrac{1}{2} d_1^2 - \tfrac{1}{2} d_4^2]\, I\\
&\quad + \tfrac{1}{6}(\tfrac{2}{3} k_{31}(0) - d_1 - d_4)(L_3 - 3C)\, I^{-1/2}\\
&\quad - \tfrac{1}{3} k_{31}(0)(L_{11} + C)\, I^{-1/2} + \tfrac{1}{2}(L_{11} + C)(d_1 + d_4)\, I^{-1/2}\\
&\quad - \tfrac{1}{12}(L_3 - 3C)^2 I^{-2},\\
A_2(2) &= (d_4 d_5 - d_1 d_2)\, I + \tfrac{1}{6}(d_2 - d_5)(3L_{11} - L_3 + 6C)\, I^{-1/2},
\end{aligned}
$$

$$A_2(3) = \tfrac{1}{3}k_{32}(1)\, I^{1/2} - \tfrac{1}{4}k_{42}(0)\, I + \tfrac{1}{9}(k_{31}(0))^2\, I$$
$$- \tfrac{2}{9}k_{31}(0)(L_3 - 3c)\, I^{-1/2} - \tfrac{1}{2}(d_2^2 + d_5^2 + d_1 d_3 + d_4 d_6)\, I$$
$$+ \tfrac{1}{6}(d_1 + d_4 - d_3 - d_6)(L_3 - 3C)\, I^{-1/2}$$
$$+ \tfrac{1}{2}(L_{11} + C)(d_3 + d_6)\, I^{-1/2}$$
$$+ \tfrac{1}{36}(L_3 - 3C)^2\, I^{-2}$$
$$A_2(4) = (d_5 d_6 - d_2 d_3)\, I + \tfrac{1}{6}(d_2 - d_5)(L_3 - 3C)\, I^{-1/2},$$
$$A_2(5) = -\tfrac{1}{2}(d_3^2 + d_6^2)\, I + \tfrac{1}{6}(d_3 + d_6)(L_3 - 3C)\, I^{-1/2}. \tag{2.14}$$

Since the k_{ij} are polynomials in δ, it follows from (1.18) that condition (C2) of Section 1 holds. We therefore get the following lemmas. *Below C_2, C_3, C_4 and C_5 will denote constants depending only on $\{f(x;\theta)\}$ and V_1, V_2 will denote non-negative constants depending on W_n which vanish for W_n^2.*

LEMMA 2.1. (a) Suppose that

$$\tfrac{1}{6}k_{32}(1) - \tfrac{1}{8}k_{42}(0)\, I^{1/2} + \tfrac{1}{18}(k_{31}(0))^2\, I^{1/2} - \tfrac{1}{9}k_{31}(0)(L_3 - 3C)\, I^{-1}$$
$$= C_2 - \tfrac{1}{2}(\gamma_{\theta_0} V)^2\, I^{7/2}, \tag{2.15}$$

where V is a constant depending on W_n and $\{f(x;\theta)\}$ which vanishes for W_n^2. Then the left side of Inequality (1.20) is $2(\gamma_{\theta_0} V I^2)^2$ and so λ_n is inferior to Rao's statistic in the sense (I) if $\gamma_{\theta_0} V \neq 0$.

(b) Suppose that

$$k_{22}(2) + (2k_{12}(1) - k_{32}(1))\, I^{1/2} + [\tfrac{1}{2}k_{42}(0) + \tfrac{1}{4}(k_{21}(0))^2$$
$$- k_{22}(0) - \tfrac{1}{18}(k_{31}(0))^2]\, I + \tfrac{1}{9}k_{31}(0)(L_3 - 3L_{11} - 6C)\, I^{-1/2}$$
$$= C_3 + g(V) - \gamma_{\theta_0}^2 V_1, \tag{2.16}$$

where g is a real-valued function. If $V = 0$ and

$$g(0) = 0, \tag{2.17}$$

then λ_n is inferior to Rao's statistic in the sense (II).

LEMMA 2.2. *Equation* (1.22) *holds iff*

$$(k_{31}(0)\, k_{21}(0) - 2k_{32}(0))\, I^{1/2} + 3k_{22}(1)$$
$$= k_{21}(0)(L_3 - 3C)\, I^{-1}. \tag{2.18}$$

Let

$$P_2(\delta, \alpha) = \phi(z - \delta I^{1/2}) \left[e_1 + \sum_1^5 \delta^j Q_j \right], \tag{2.19}$$

where the Q_j are free from δ and are polynomials in z. Assume that

$$\begin{aligned} &k_{12}(r)=0 \quad \textit{if} \quad r\geqslant 4; \qquad k_{22}(r)=0 \quad \textit{if} \quad r\geqslant 3 \\ &k_{32}(r)=0 \quad \textit{if} \quad r\geqslant 2; \qquad k_{42}(r)=0 \quad \textit{if} \quad r\geqslant 1. \end{aligned} \tag{2.20}$$

It is now easy to verify from (2.1) that

$$Q_5=-(\bar{K})^2/2=-(3L_{11}+L_3)^2\, I^{-1/2}/72, \tag{2.21}$$

$$\begin{aligned} Q_4&=\tfrac{1}{2}z\bar{K}(k_{11}(2)-\tfrac{3}{2}k_{21}(1)\, I^{1/2}+\tfrac{5}{6}k_{31}(0)\, I) \\ &=[(9L_{11}^2-L_3^2)+6C(3L_{11}+L_3)]\, zI^{-1}/72, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \bar{K}&=k_{11}(2)-\tfrac{1}{2}k_{21}(1)\, I^{1/2}+\tfrac{1}{6}k_{31}(0)\, I, \\ Q_3&=\tfrac{1}{6}b_1[6k_{11}(2)-3k_{21}(1)\, I^{1/2}+k_{31}(0)\, I]\, I^{1/2}+k_{12}(3) \\ &\quad -\tfrac{1}{2}[k_{22}(2)+(k_{11}(1))^2+2k_{11}(0)\, k_{11}(2)]\, I^{1/2} \\ &\quad +zk_{11}(1)\, k_{11}(2)+\tfrac{1}{6}[k_{32}(1)+3k_{21}(0)\, k_{11}(1) \\ &\quad +3k_{21}(1)\, k_{11}(0)]\, I+\tfrac{1}{2}(z^2-1)\, k_{21}(1)\, k_{11}(2) \\ &\quad -z[k_{21}(0)\, k_{11}(2)+k_{21}(1)\, k_{11}(1)]\, I^{1/2} \\ &\quad -\tfrac{1}{8}(z^2-1)[3(k_{21}(1))^2+4k_{11}(2)\, k_{31}(0)]\, I^{1/2} \\ &\quad +z[3k_{21}(0)\, k_{21}(1)+2k_{11}(1)\, k_{31}(0)]\, I \\ &\quad -\tfrac{1}{24}[k_{42}(0)+3(k_{21}(0))^2+4k_{11}(0)\, k_{31}(0)]\, I^{3/2} \\ &\quad +\tfrac{1}{2}(z^2-1)\, k_{21}(1)\, k_{31}(0)\, I-\tfrac{1}{3}zk_{21}(0)\, k_{31}(0)\, I^{3/2} \\ &\quad -\tfrac{5}{36}(z^2-1)(k_{31}(0))^2\, I^{3/2}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} Q_3(0)&=k_{12}(3)-\tfrac{1}{2}k_{22}(2)\, I^{1/2}+\tfrac{1}{6}k_{32}(1)\, I-\tfrac{1}{24}k_{42}(0)\, I^{3/2} \\ &\quad -\tfrac{1}{6}(3L_{11}+L_3)\, d_1-\tfrac{1}{72}(L_3-3C)(6L_{11}+L_3+3C)\, I^{-3/2}, \\ Q_3(1)&=-\tfrac{1}{6}(3L_{11}+L_3)\, d_2, \\ Q_3(2)&=(k_{11}(2)-\tfrac{3}{4}k_{21}(1)\, I^{1/2}+\tfrac{1}{3}k_{31}(0)\, I)(\tfrac{1}{2}k_{21}(0)-\tfrac{1}{3}k_{31}(0)\, I^{1/2}) \\ &=\tfrac{1}{72}(6L_{11}+L_3)\, L_3 I^{-3/2}. \end{aligned} \tag{2.23}$$

Assume that

$$Q_3(0) \textit{ is a constant depending only on } \{f(x;\theta)\}. \tag{2.24}$$

We therefore get the following lemma. *Below g_1 and g_2 will denote real-valued functions which vanish at* 0.

LEMMA 2.3. (a) *Under the assumption of Lemma* 2.1(a), *the left side of Inequality* (1.23) *is* $\frac{1}{2}|\delta|(\gamma_{\theta_0}V)^2 I^{7/2}$ *and so* λ_n *is inferior to Rao's statistic in the sense* (III) *provided* $\gamma_{\theta_0}V \neq 0$.

(b) *Suppose that* (2.25) *to* (2.27) *below hold.*

$$V=0 \Rightarrow (2.18) \text{ holds}, \tag{2.25}$$

$$\begin{aligned}&\tfrac{1}{2}k_{22}(2)-\tfrac{1}{3}k_{32}(1)\,I^{1/2}+\tfrac{1}{8}k_{42}(0)\,I-\tfrac{1}{6}k_{31}(0)(L_{11}+C)\,I^{-1/2}\\&\quad=C_4+g_1(v),\end{aligned} \tag{2.26}$$

$$\begin{aligned}&k_{12}(1)-\tfrac{1}{6}k_{32}(1)+[\tfrac{1}{8}k_{42}(0)+\tfrac{1}{8}(k_{21}(0))^2\\&\qquad-\tfrac{1}{2}k_{22}(0)-\tfrac{1}{36}(k_{31}(0))^2]\,I^{1/2}+\tfrac{1}{18}k_{31}(0)(L_3-3C)\,I^{-1}\\&\quad=C_5+g_2(v)-\gamma_{\theta_0}^2V_2.\end{aligned} \tag{2.27}$$

If $V=0$, *then the left side of* (1.24) *is* $\gamma_{\theta_0}^2V_2$ *and so* λ_n *is inferior to Rao's statistic in the sense* (IV) *provided* $\gamma_{\theta_0}V_2 \neq 0$.

(c) *Suppose that*

$$V=0 \qquad \text{and} \qquad V_2=0 \Rightarrow Q_2(0)=0. \tag{2.28}$$

Then $P_2^{\lambda}(\delta, \alpha)$ *and* $P_2(\delta, \alpha)$ *are identical.*

3. PERTURBED MLE'S

Consider the family $\mathscr{F}$ of statistics described in Section 1 and let $\{\lambda_n\}$ be in $\mathscr{F}$. Our aim is to show that the coditions of Section 2 are satisfied for λ_n, provided certain regularity conditions on $\{f(x;\theta)\}$ hold. Let $K_{i,n}$ be the ith "approximate cumulant" under θ_n of W_n and *assume that*

$$\begin{aligned}&K_{i,n}=k_{i0}+n^{-1/2}k_{i1}+n^{-1}k_{i2}+o(n^{-1}), \qquad i\geqslant 1,\\&k_{10}=\delta I^{1/2},\, k_{20}=1,\, k_{30}=k_{40}=k_{41}=0,\\&k_{ij}=0 \qquad \text{for} \quad j=0,1,2 \qquad \text{and} \qquad i\geqslant 5,\end{aligned} \tag{3.1}$$

the k_{ij} *being free from n. Assume that in a neighbourhood of* θ_0,

$$m_1(\theta)\equiv 0, \qquad m_2(\theta)+I(\theta)\equiv 0, \tag{3.2}$$

and that the Edgeworth expansion of the distribution funciton under θ_n of W_n *obtained by the formal delta method is valid* (see [1, 2] in this connection).

Appendix 2 gives the expressions for this $K_{i,n}$; to that end, one first replaces the H_i by their Taylor's expansions involving the Δ_i, where

$$\begin{aligned} \Delta_i &= n^{-1/2}\left(\sum_j D^i \log f(x_j;\theta_0) - nl_i^*\right) \\ l_i^* &= m_i(\theta_n), \qquad i \geqslant 1, \end{aligned} \tag{3.3}$$

and then rewrites W_n as follows.

$$\begin{aligned} W_n = {} & \delta I^{1/2} + \Delta_1 I^{-1/2} + n^{-1/2}[u_1\delta\Delta_2 + u_2\delta^2 + u_3\delta\Delta_1 \\ & + u_4\delta + v_1\Delta_1\Delta_2 + v_2\Delta_1^2 + v_3\Delta_2^2 + v_4\Delta_1 + v_5\Delta_2 \\ & + n^{-1}[x_1\delta\Delta_1\Delta_2 + x_2\delta\Delta_1^2 + x_3\delta\Delta_1\Delta_3 + x_4\delta^2\Delta_3 \\ & + x_5\delta^2\Delta_1 + x_6\delta^3 + x_7\delta\Delta_2^2 + x_8\delta^2\Delta_2 + x_9\delta\Delta_2\Delta_3 \\ & + x_{10}\delta^2 + x_{11}\delta\Delta_2 + x_{12}\delta\Delta_3 + x_{13}\delta\Delta_3^2 + x_{14}\delta\Delta_1 \\ & + x_{15}\delta + y_1\Delta_1\Delta_2^2 + y_2\Delta_1^2\Delta_2 + y_3\Delta_1^3 + y_4\Delta_1^2\Delta_3 \\ & + y_5\Delta_1\Delta_3^2 + y_6\Delta_2^3 + y_7\Delta_1\Delta_2 + y_8\Delta_1^2 + y_9\Delta_2^2 \\ & + y_{10}\Delta_1 + y_{11}\Delta_2] + o(n^{-1}) \end{aligned} \tag{3.4}$$

(for each real δ), the constants u_i and x_i being bounded functions of n; Appendix 1 gives the expressions for the u_i and x_i. It is then easy to verify that (2.1), (2.5), (2.6) and (2.20) hold. Appendix 2 yields

$$\begin{aligned} k_{11}(1) &= v_4 I + v_5(m_2' - L_{001}), \\ k_{11}(2) &= v_1(m_2' - L_{001})\, I + v_2 I^2 + v_3(m_2' - L_{001})^2 + (\tfrac{1}{2}L_{001} + I')\, I^{-1/2}, \\ k_{21}(0) &= 2(v_4 I + v_5 L_{11})\, I^{-1/2}, \\ k_{21}(1) &= 2v_1(m_2' - L_{001} + L_{11})\, I^{1/2} + 2v_2 I^{3/2} \\ &\quad + 2v_3 L_{11}(m_2' - L_{001})\, I^{-1/2} + (I' - 2L_{11})\, I^{-1}, \\ k_{31}(0) &= 6(v_1 L_{11} + v_2 I + v_3 L_{11}^2 I^{-1}) + L_3 I^{-3/2}; \end{aligned} \tag{3.5}$$

here and below the primes stand for differentiation with respect to θ at $\theta = \theta_0$. Assume that

$$m_2' = L_{001} + L_{11}; \tag{3.6}$$

then (2.2) to (2.4) hold.

To establish (2.15) and (2.16), first note that

$$\begin{aligned}
k_{32}(1) = {} & (L'_3 - 3L_{21} - 3I^2)\, I^{-3/2} + 3v_1[L_{21} + 3I^2 \\
& - 2(L_{101} + L_{02} - L'_{11}) + L_{11}(L_3 - 4L_{11} + 2I')\, I^{-1}] \\
& + 6v_2(L_3 + 2I' - 4L_{11}) + 6v_3 L_{11}[3I^2 + L_{21} \\
& - 2(L_{101} + L_{02} - L'_{11})]\, I^{-1} + 6v_1^2[I(L_{02} - I^2) \\
& + 3L_{11}^2]\, I^{1/2} + 24v_2^2 I^{5/2} + 24v_3^2 L_{11}^2(L_{02} - I^2)\, I^{-1/2} \\
& + 48v_1 v_2 L_{11} I^{3/2} + 24v_1 v_3[I(L_{02} - I^2) + L_{11}^2]\, L_{11} I^{-1/2} \\
& + 48v_2 v_3 L_{11}^2 I^{1/2} + 6[3y_1 L_{11}^2 + 3y_2 L_{11} I \\
& + y_4(m'_3 - L_{0001} + 2L_{101})\, I + y_5(2m'_3 - 2L_{0001} + L_{101})\, L_{101} \\
& + 3y_6 L_{11}^3 I^{-1}], \\
k_{42}(0) = {} & (L_4 - 3I^2)\, I^{-2} + 12v_1(L_{21} + I^2 + L_3 L_{11} I^{-1})\, I^{-1/2} \\
& + 24v_2 L_3 I^{-1/2} + 24v_3(L_{21} + I^2)\, L_{11} I^{-3/2} \\
& + 12v_1^2[I(L_{02} - I^2) + 3L_{11}^2] + 48v_2^2 I^2 \\
& + 48v_3^2(L_{02} - I^2)\, L_{11}^2 I^{-1} + 96v_1 v_2 L_{11} I \\
& + 48v_1 v_3[I(L_{02} - I^2) + L_{11}^2]\, L_{11} I^{-1} + 96v_2 v_3 L_{11}^2 \\
& + 24[y_1 L_{11}^2 + y_2 L_{11} I + y_3 I^2 + y_4 L_{101} I \\
& + y_5 L_{101}^2 + y_6 L_{11}^3 I^{-1}]\, I^{-1/2}, \\
k_{12}(1) = {} & v_1(L'_{11} - L_{101} - L_{02} + I^2) + v_2(I' - 2L_{11}) \\
& + v_3(L'_{02} + 2m'_2 I - 2L_{011} - 2L_{001} I) + y_1[2L_{11}(m'_2 - L_{001}) \\
& + (L_{02} - I^2)\, I] + y_2(2L_{11} + m'_2 - L_{001})\, I + 3y_3 I^2 \\
& + y_4(m'_3 - L_{0001} + 2L_{101})\, I + y_5[2(m'_3 - L_{0001})\, L_{101} \\
& + (L_{0002} - L_{001}^2)\, I] + 3y_6(L_{02} - I^2)(m'_2 - L_{001}) \\
& + y_{10} I + y_{11}(m'_2 - L_{001}), \\
k_{22}(0) = {} & 2[v_1(L_{21} + I^2) + v_2 L_3 + v_3(L_{12} + 2L_{11} I) \\
& + y_1(I(L_{02} - I^2) + 2L_{11}^2) + 3y_2 L_{11} I + 3y_3 I^2 \\
& + 3y_4 L_{101} I + y_5(I(L_{002} - L_{001}^2) + 2L_{101}^2) \\
& + 3y_6(L_{02} - I^2)\, L_{11} + y_{10} I + y_{11} L_{11}]\, I^{-1/2} \\
& + v_1^2(I(L_{02} - I^2) + L_{11}^2) + 2v_2^2 I^2 + 2v_3^2(L_{02} - I^2)^2 \\
& + v_4^2 I + v_5^2(L_{02} - I^2) + 4v_1 v_2 L_{11} I + 4v_1 v_3 L_{11}(L_{02} - I^2) \\
& + 4v_2 v_3 L_{11}^2 + 2v_4 v_5 L_{11},
\end{aligned}$$

$$
\begin{aligned}
k_{22}(2) = {} & \tfrac{1}{2}(I'' - 4L'_{11} + 2L_{101} + 2L_{02} - 2I^2)\, I^{-1} \\
& + v_1[(4I' + L_{001} - 4L_{11})\, L_{11} I^{-1/2} \\
& + (2L'_{11} - 2m'_3 - 2L_{101} + m''_2 + L_{0001} - 2L_{02} + 2I^2)\, I^{1/2}] \\
& + 2v_2(4I' + L_{001} - 4L_{11})\, I^{1/2} \\
& + 2v_3[2(L'_{11} - L_{101} - L_{02} + I^2 - m'_3) + m''_2 + L_{0001}]\, L_{11} I^{-1/2} \\
& + v_1^2[I(L_{02} - I^2) + 3L_{11}^2] + 4v_2^2 I^3 \\
& + 4v_3^2(L_{02} - I^2)\, L_{11}^2 + 8v_1 v_2 L_{11} I^2 \\
& + 4v_1 v_3[I(L_{02} - I^2) + L_{11}^2]\, L_{11} + 8v_2 v_3 L_{11}^2 I \\
& + 6y_1 L_{11}^2 I^{1/2} + 6y_2 L_{11} I^{3/2} + 6y_3 I^{5/2} \\
& + 2y_4[L_{101} + 2(m'_3 - L_{0001})]\, I^{3/2} + 2y_5(m'_3 - L_{0001}). \\
& (m'_3 - L_{0001} + 2L_{101})\, I^{1/2} + 6y_6 L_{11}^3 I^{-1/2}.
\end{aligned}
$$

The left side of (2.15) *is*

$$
\begin{aligned}
& \tfrac{1}{24}[L_4 - 3I^2 + 4(L'_3 - 3L_{21} - L_4)]\, I^{-3/2} - \tfrac{1}{18} L_3(L_3 - 6C)\, I^{-5/2} \\
& \quad - \tfrac{1}{2}(\gamma_{\theta_0} V)^2 I^{7/2} + V(L'_{11} - L_{101} - L_{02} - L_{21} + CL_{11} I^{-1}) \\
& \quad + (y_4 I + 2y_5 L_{101})(m'_3 - L_{0001} - L_{101}),
\end{aligned}
$$

V being as in (1.10), and so (2.15) holds *under the assumptions that*

$$m'_3 = L_{0001} + L_{101}, \tag{3.7}$$

$$L'_{11} - L_{101} - L_{02} - L_{21} + CL_{11} I^{-1} = 0. \tag{3.8}$$

Similarly,

$$
\begin{aligned}
& \tfrac{1}{36}(k_{31}(0))\, I^{1/2} - \tfrac{1}{18} k_{31}(0)(L_3 - 3C)\, I^{-1} + k_{12}(1) \\
& \quad - \tfrac{1}{2} k_{22}(0)\, I^{1/2} + \tfrac{1}{8}(k_{21}(0))^2 I^{1/2} \\
& = -\tfrac{1}{36} L_3(L_3 - 6C)\, I^{-5/2} - \tfrac{1}{2}(\gamma_{\theta_0} V)^2 I^{7/2} - \tfrac{1}{2}\gamma_{\theta_0}^2 V_1 \\
& \quad + v_1(L'_{11} - L_{101} - L_{02} - L_{21} + CL_{11} I^{-1}) \\
& \quad + v_3(L'_{02} - 2L_{011} - L_{12} + CL_{11}^2 I^{-1}) \\
& \quad + (y_4 I + 2y_5 L_{101})(m'_3 - L_{0001} - L_{101}),
\end{aligned}
$$

$$V_1 = (v_5^2 + 2\gamma_{\theta_0}^2 v_3^2 I^2)\, I^{5/2},$$

$$\begin{aligned}
&\tfrac{1}{2}k_{22}(2) - \tfrac{1}{3}k_{32}(1)\, I^{1/2} + \tfrac{1}{8}k_{42}(0)\, I - \tfrac{1}{6}k_{31}(0)(L_{11} + C)\, I^{-1/2} \\
&= [\tfrac{1}{4}(I'' - 4L'_{11} + 2L_{101} + 2L_{02} - 2I^2) - \tfrac{1}{3}(L'_3 - 3L_{21} - 3I^2) \\
&\quad + \tfrac{1}{8}(L_4 - 3I^2)]\, I^{-1} - \tfrac{1}{6}(L_{11} + C)\, L_3 I^{-2} + \tfrac{1}{2}g(V),
\end{aligned}$$

$$g(V) = \gamma_{\theta_0}^2 V I^{5/2} - C_1 V I^{1/2}.$$

Equations (2.15), (3.7) and (3.8) now yield (2.16) and (2.17), *under the further assumption that*

$$L'_{02} - 2L_{011} - L_{12} + C L_{11}^2 I^{-2} = 0; \tag{3.9}$$

moreover,

$$\begin{aligned}
C_3 = {} & (\tfrac{1}{2}I'' - 2L'_{11} + L_{101} + L_{02} - I^2 - \tfrac{15}{24}(L_4 - 3I^2) \\
& + 3L_{21} + L_4 - L'_3)\, I^{-1} - \tfrac{1}{18}L_3(6L_{11} + 12C - L_3)\, I^{-2}.
\end{aligned}$$

We have also verified (2.26) and (2.27) with $g_1 = \frac{1}{2}g$, $g_2 \equiv 0$ and $V_2 = \frac{1}{2}V_1$.

To verify (2.24), one need only note that

$$\begin{aligned}
k_{12}(3) = {} & -\tfrac{1}{6}(L_{0001} + 3m''_2 - 3m'_3)\, I^{-1/2} \\
& + \tfrac{1}{2}v_1[(m''_2 + L_{0001} - 2m'_3)\, I - (m'_2 - L_{0001})(2m'_2 - L_{001})] \\
& - v_2(2m'_2 - L_{001})\, I + v_3 L_{11}(m''_2 + L_{0001} - 2m'_3) \\
& + y_1 L_{11}^2 I + y_2 L_{11} I^2 + y_3 I^3 + y_4 L_{101} I^2 \\
& + y_5 L_{101}^2 I + y_6 L_{11}^3,
\end{aligned}$$

and use (3.7). Condition (2.25) follows from the fact that

$$\begin{aligned}
&\tfrac{1}{6}k_{31}(0)\, k_{21}(0)\, I^{1/2} - \tfrac{1}{3}k_{32}(0)\, I^{1/2} + \tfrac{1}{2}k_{22}(1) - \tfrac{1}{6}k_{21}(0)(L_3 - 3C)\, I^{-1} \\
&= v_5[I^{-1/2}(L'_{11} - L_{101} - L_{02} - L_{21} + C L_{11} I^{-1}) - I^3 \gamma_{\theta_0}^2 V];
\end{aligned}$$

one may note here that Eq. (1.22) *holds iff* $\gamma_{\theta_0}^2 v_5 V = 0$ (use Lemma 2.2 and (3.8)). To verify (2.28), first note that

$$\begin{aligned}
k_{12}(2) = {} & v_4(I' + \tfrac{1}{2}L_{001}) + \tfrac{1}{2}v_5(m''_2 + L_{0001} - 2m'_3) \\
& + y_7 L_{11} I + y_8 I^2 + y_9 L_{11}^2, \\
k_{22}(1) = {} & 2[v_4(I' - 2L_{11}) + v_5(L'_{11} - L_{101} - L_{02} + I^2) \\
& + 2y_7 L_{11} I + 2y_8 I^2 + 2y_9 L_{11}^2]\, I^{-1/2} \\
& + 2[2v_1 v_4 L_{11} I + v_1 v_5(I(L_{02} - I^2) + L_{11}^2) \\
& + v_2 v_4 I^2 + (v_2 v_5 I + v_3 v_4 L_{11} + v_3 v_5(L_{02} - I^2))\, L_{11}],
\end{aligned}$$

$$k_{32}(0) = 3[v_4 L_3 + v_5(L_{21} + I^2) + 2y_7 L_{11} I + 2y_8 I^2 + 2y_9 L_{11}^2]\, I^{-1}$$
$$+ 6[2v_1 v_4 L_{11} I + v_1 v_5(I(L_{02} - I^2) + L_{11}^2) + 2v_2 v_4 I^2$$
$$+ 2v_2 v_5 L_{11} I + 2v_3 v_4 L_{11}^2 + 2v_3 v_5 L_{11}(L_{02} - I^2)]\, I^{-1/2};$$

it then follows that

$$Q_2(0) = \tfrac{1}{2}(\gamma_{\theta_0}^2 I^2 - C_1)\, v_5, \tag{3.10}$$

which implies the desired result. Equation (1.9) now follows, since

$$P_{\theta_n}(W_n > z + n^{-1/2} b_1 + n^{-1} c_1) - P_{\theta_n}(W_n^2 > z + n^{-1/2} b_1^2 + n^{-1} c_1^2)$$
$$= \phi(z - \delta I^{1/2})[\gamma_{\theta_0}^2 \delta\{\tfrac{1}{2} V^2 z^2 I^{7/2} + V v_5 z I^3 + \gamma_{\theta_0}^2 v_3^2 I^{9/2}$$
$$+ \tfrac{1}{2} v_5^2 I^{5/2}\} + \delta^2\{\tfrac{1}{2} C_1 V z I^{1/2} - \tfrac{1}{2}\gamma_{\theta_0}^2 V z I^{5/2}$$
$$+ \tfrac{1}{2} v_5 (C_1 - \gamma_{\theta_0}^2 I^2)\}]. \tag{3.11}$$

4. Some Modifications of λ_n^3 and λ_n^2

Following a suggestion of Efron and Hinkley [5], Skovgaard [12] studies in Section 6 the conditional null distributions of λ_n^1, λ_n^3 and λ_n^4, where

$$\lambda_n^4 = (\hat{\theta} - \theta_0)^2 L, \qquad L = \Sigma D^2 \log f(x_i; \hat{\theta}),$$

given the normalised Efron–Hinkley ancillary statistic. He comments that "a possibility would be to compare the (asymptotic) power of the tests, but a uniform superiority of any of these could hardly be expected." Hayakawa and Puri [8] obtain the expansions of λ_n^5 and other statistics, where

$$\lambda_n^5 = n(\hat{\theta} - \theta_0)^2 I(\theta_0),$$

and state that (see p. 97) λ_n^5 "is more powerful than other statistics for parameters of the specified structure and in the region of certain alternative." We propose the following additional analogues of λ_n^3 and λ_n^2.

$$\lambda_n^6 = -(\hat{\theta} - \theta_0)^2 \sum D^2 \log f(x_i; \theta_0),$$

$$\lambda_n^7 = \left(\sum D \log f(x_i; \theta_0)\right)^2 \Big/ L,$$

$$\lambda_n^8 = \left(\sum D \log f(x_i; \theta_0)\right)^2 (nI(\hat{\theta}))^{-1},$$

$$\lambda_n^9 = -\left(\sum D \log f(x_i; \theta_0)\right)^2 \Big/ \left(\sum D^2 \log f(x_i; \theta_0)\right).$$

Below we shall compare the (local) powers of the two-sided tests based on the W_n^i, $i=4, ..., 9$, under the assumptions of Section 3 and show that these statistics are inferior to λ_n^2 in the sense of (I) and (III).

Proceeding as in Section 3 of [3], it is straightforward to verify that the above statistics belong to the family of statistics considered in [4]. Hence the coefficients of n^{-1} in the powers of the corresponding tests will depend only on their respective v_1. The values of $v_1 I^{3/2}$ for the λ_n^i, $i=1, ..., 9$, are respectively $\frac{1}{2}$, 0, 1, $\frac{1}{2}$, 1, $\frac{1}{2}$, $\frac{1}{2}$, 0, $\frac{1}{2}$. We can, therefore conclude that (up to $o(n^{-1})$)

(a) the $P^i_{n,\delta,\alpha}$, $i=1, 4, 6, 7, 9$, are identical;

(b) $P^2_{n,\delta,\alpha} \equiv P^8_{n,\delta,\alpha}$;

(c) $P^3_{n,\delta,\alpha} \equiv P^5_{n,\delta,\alpha}$.

Thus the performance of λ_n^5 (as a test statistic), though simpler than λ_n^3, is identical with that of λ_n^3; the modified versions of Rao's statistic using observed Fisher information in place of expected Fisher information is inferior to Rao's statistic; similar modifications in Wald's statistic yield better test statistics compared to Wald's statistic, although they are inferior to Rao's statistic. It appears that in case of test statistics, the observed Fisher information has the same effect as the localized Fisher information, $-\sum D^2 \log f(x_i; \theta_0)$, on their local powers.

Appendix 1

$$
\begin{aligned}
u_1 &= -I^{-1/2} - v_1 l_2^* + 2v_3(m_2' - l_3^*),\\
u_2 &= -m_2' I^{-1/2} + \tfrac{1}{2} l_3^* I^{-1/2} - v_1 l_2^*(m_2' - l_3^*)\\
&\qquad + v_2 l_2^{*2} + v_3(m_2' - l_3^*)^2,\\
u_3 &= v_1(m_2' - l_3^*) - 2v_2 l_2^*,\\
u_4 &= -v_4 l_2^* + v_5(m_2' - l_3^*),\\
x_1 &= -2v_2 + 2y_1(m_2' - l_3^*) - 2y_2 l_2^*,\\
x_2 &= y_2(m_2' - l_3^*) - 3y_3 l_2^* + y_4(m_3' - l_4^*),\\
x_3 &= -v_1 - 2y_4 l_2^* + 2y_5(m_3' - l_4^*),\\
x_4 &= \tfrac{1}{2} I^{-1/2} + v_1 l_2^* - 2v_3(m_2' - l_3^*) + y_4 l_2^{*2} - 2y_5 l_2^*(m_3' - l_4^*),\\
x_5 &= \tfrac{1}{2} v_1(m_2'' + l_4^*) + v_2 l_3^* + y_1(m_2' - l_3^*)^2 - 2y_2 l_2^*(m_2' - l_3^*)\\
&\qquad + 3y_3 l_2^{*2} - 2y_4 l_2^*(m_3' - l_4^*) + y_5(m_3' - l_4^*)^2,
\end{aligned}
$$

$$x_6 = -\tfrac{1}{2}m_2'' I^{-1/2} - \tfrac{1}{6}l_4^* I^{-1/2} - \tfrac{1}{2}v_1(m_2''+l_4^*)\, l_2^* + \tfrac{1}{2}v_1 l_3^*(m_2'-l_3^*)$$
$$- v_2 l_2^* l_3^* + v_3(m_2'-l_3^*)(m_2''+l_4^*) - y_1 l_2^*(m_2'-l_3^*)^2$$
$$+ y_2 l_2^{*2}(m_2'-l_3^*) - y_3 l_2^{*3} + y_4 l_2^{*2}(m_3'-l_4^*)$$
$$- y_5 l_2^*(m_3'-l_4^*)^2 + y_6(m_2'-l_3^*)^3,$$
$$x_7 = -v_1 - y_1 l_2^* + 3y_6(m_2'-l_3^*),$$
$$x_8 = \tfrac{1}{2}v_1 l_3^* - v_1(m_2'-l_3^*) + 2v_2 l_2^* + v_3(m_2''+l_4^*)$$
$$- 2y_1 l_2^*(m_2'-l_3^*) + y_2 l_2^{*2} + 3y_6(m_2'-l_3^*)^2,$$
$$x_9 = -2v_3,\ x_{12} = -v_5,\ x_{13} = -y_5 l_2^*,$$
$$x_{10} = \tfrac{1}{2}v_4 l_3^* + \tfrac{1}{2}v_5(m_2''+l_4^*) - y_7 l_2^*(m_2'-l_3^*)$$
$$+ y_8 l_2^{*2} + y_9(m_2'-l_3^*)^2,$$
$$x_{11} = -v_4 - y_7 l_2^* + 2y_9(m_2'-l_3^*),$$
$$x_{14} = y_7(m_2'-l_3^*) - 2y_8 l_2^*,$$
$$x_{15} = -y_{10} l_2^* + y_{11}(m_2'-l_3^*).$$

Appendix 2

Define l_{ijk} as on page 244 of [3]. The following equalities are correct up to $o(n^{-1})$, unless otherwise stated.

$$K_{1,n} = \delta I^{1/2} + n^{-1/2}[u_2\delta^2 + u_4\delta + v_1 l_{11} + v_2 l_2 + v_3 l_{02}]$$
$$+ n^{-1}[\delta(x_1 l_{11} + x_2 l_2 + x_3 l_{101} + x_6\delta^2 + x_7 l_{02}$$
$$+ x_9 l_{011} + x_{10}\delta + x_{13} l_{002} + x_{15})$$
$$+ y_7 l_{11} + y_8 l_2 + y_9 l_{02}],$$
$$K_{2,n} = l_2 I^{-1} + 2n^{-1/2}[(u_1 l_{11} + u_3 l_2)\,\delta + v_4 l_2 + v_5 l_{11}]\, I^{-1/2}$$
$$+ n^{-1}[2\{\delta^2(x_4 l_{101} + x_5 l_2 + x_8 l_{11}) + \delta(x_{11} l_{11}$$
$$+ x_{12} l_{101} + x_{14} l_2) + v_1 l_{21} + v_2 l_3 + v_3 l_{12}$$
$$+ y_1(l_2 l_{02} + 2l_{11}^2) + 3y_2 l_2 l_{11} + 3y_3 l_2^2$$
$$+ 3y_4 l_2 l_{101} + y_5(l_2 l_{002} + 2l_{101}^2).$$
$$+ 3y_6 l_{02} l_{11} + y_{10} l_2 + y_{11} l_{11}\}\, I^{-1/2}$$
$$+ \delta^2(u_1^2 l_{02} + u_3^2 l_2 + 2u_1 u_3 l_{11})$$

$$
\begin{aligned}
&+2\delta(u_1 v_4 l_{11} + u_1 v_5 l_{02} + u_3 v_4 l_2 + u_3 v_5 l_{11}) \\
&+ v_1^2(l_2 l_{02} + l_{11}^2) + 2v_2^2 l_2^2 + 2v_3^2 l_{02}^2 + v_4^2 l_2 + v_5^2 l_{02} \\
&+ 4v_1 v_2 l_2 l_{11} + 4v_1 v_3 l_{02} l_{11} + 4v_2 v_3 l_{11}^2 + 2v_4 v_5 l_{11}],
\end{aligned}
$$

$$
\begin{aligned}
K_{3,n} = {} & n^{-1/2}[l_3 I^{-3/2} + 6(v_1 l_2 l_{11} + v_2 l_2^2 + v_3 l_{11}^2) I^{-1}] \\
&+ 3n^{-1}[\{\delta(u_1 l_{21} + u_3 l_3 + 2x_1 l_2 l_{11} \\
&+ 2x_2 l_2^2 + 2x_3 l_2 l_{101} \\
&+ 2x_7 l_{11}^2 + 2x_9 l_{11} l_{101} + 2x_{13} l_{101}^2) + v_4 l_3 \\
&+ v_5 l_{21} + 2y_7 l_2 l_{11} + 2y_8 l_2^2 + 2y_9 l_{11}^2\} I^{-1} \\
&+ \{\delta(2u_1 v_1(l_2 l_{02} + l_{11}^2) + 4u_1 v_2 l_2 l_{11} \\
&+ 4u_1 v_3 l_{02} l_{11} + 4u_3 v_1 l_2 l_{11} + 4u_3 v_2 l_2^2 + 4u_3 v_3 l_{11}^2) \\
&+ 4v_1 v_4 l_{11} l_2 + 2v_1 v_5(l_2 l_{02} + l_{11}^2) + 4v_2 v_4 l_2^2 \\
&+ 4v_2 v_5 l_2 l_{11} + 4v_3 v_4 l_{11}^2 + 4v_3 v_5 l_{02} l_{11}\} I^{-1/2}],
\end{aligned}
$$

$$
\begin{aligned}
K_{4,n} = {} & n^{-1}[(l_4 - 3l_2^2) I^{-1} + 12v_1(l_{21} - l_{11} l_3 I^{-1}) I^{-1/2} \\
&+ 24v_2 l_3 I^{-1/2} + 24v_3 l_{21} l_{11} I^{-3/2} + 12v_1^2(l_2 l_{02} + 3l_{11}^2) \\
&+ 48v_2^2 l_2^2 + 48v_3^2 l_{02} l_{11}^2 I^{-1} + 96v_1 v_2 l_2 l_{11} \\
&+ 48v_1 v_3(l_{11} l_{02} + l_{11}^3 I^{-1}) + 96v_2 v_3 l_{11}^2 \\
&+ 24(y_1 l_{11}^2 + y_2 l_2 l_{11} + y_3 l_2^2 + y_4 l_2 l_{101} \\
&+ y_5 l_{101}^2 + y_6 l_{11}^3 I^{-1}) I^{-1/2}].
\end{aligned}
$$

To get the above expressions of $K_{i,n}$, the following moments under θ_n, in addition to those given in p. 245 of [3], are needed.

$$
\begin{aligned}
E\Delta_1 \Delta_2 \Delta_3 &= O(n^{-1/2}), \\
E\Delta_1^2 \Delta_2 \Delta_3 &= l_2 l_{011} + 2l_{11} l_{101} + O(n^{-1}), \\
E\Delta_1^3 \Delta_2 \Delta_3 &= O(n^{-1/2}), \\
E\Delta_1^3 \Delta_2^3 &= 9l_2 l_{11} l_{02} + 6l_{11}^3 + O(n^{-1}).
\end{aligned}
$$

REFERENCES

[1] BHATTACHARYA, R. N. (1985). Some recent results on Cramér–Edgeworth expansions with applications. In *Multivariate Analysis VI* (P. R. Krishnaiah, Ed.), North-Holland, Amsterdam.

[2] BHATTACHARYA, R. N., AND GHOSH, J. K. (1978). On the validity of the formal Edgeworth expansion. *Ann. Statist.* **6** 434–451. Correction, ibid, **8** 1399.

[3] CHANDRA, T. K., AND JOSHI, S. N. (1983). Comparison of the likelihood ratio, Rao's and Wald's tests and a conjecture of C. R. Rao. *Sankhyā Ser. A* **45** 226–246.

[4] CHANDRA, T. K., AND MUKERJEE, R. (1984). On the optimality of Rao's statistic. *Comm. in Statist. A-Theory Methods* **13** 1507–1515.

[5] EFRON, B. AND HINKLEY, D. V. (1978). Assessing the accuracy of the maximum likelihood estimator: Observed versus expected Fisher information. *Biometrika* **65** 457–482.

[6] GHOSH, J. K., SINHA, B. K., AND JOSHI, S. N. (1982). Expansions for posterior probability and integrated Bayes risk. In *Statistical Decision Theory and Related Topics III* (S. S. Gupta and J. O. Berger, Eds). Vol. 1, pp. 403–456. Academic Press, Orlando FL.

[7] HARRIS, P. AND PEERS, H. W. (1980). The local power of the efficient scores test statistic. *Biometrika* **67** 525–529.

[8] HAYAKAWA, T., AND PURI, M. L. (1985). Asymptotic expansions of the distributions of some test statistics. *Ann. Inst. Statist. Math.* **37**, Part *A* 95–108.

[9] PEERS, H. W. (1971). Likelihood ratio and associated test criteria. *Biometrika* **58** 577–587.

[10] RAO, C. R. (1962). Efficient estimates and optimum inference procedures in large samples (with discussions). *J. Roy. Statist. Soc. Ser. B* **24** 46–72.

[11] RAO, C. R. (1965). *Linear Statistical Inference and Its Applications.* Wiley, New York.

[12] SKOVGAARD, IB. M. (1985). A second-order investigation of asymptotic ancillarity. *Ann. Statist.* **13** 534–551.